II. *Description of a new instrument for performing mechanically the involution and evolution of numbers. By* Peter M. Roget, *M. D. Communicated by* William Hyde Wollaston, *M. D. Sec. R. S.*

Read November 17, 1814.

To abridge that species of mental labour which is required in conducting arithmetical computations, has been the professed object of a variety of mechanical contrivances. But the greater number of arithmetical machines, as they have been called, are more ingenious than really useful, and have been recorded more as objects of curiosity, than as admitting of convenient or ready application in the actual practice of arithmetic. The machine invented by Pascal, and others constructed on the same principle, were, strictly speaking, limited to the simpler operations of addition and subtraction, and were incapable of being applied to the finding of products or quotients in any other way than by effecting a number of successive additions or subtractions. Still less did they aim at the immediate performance of the higher operations of involution, which, even by the most compendious methods of arithmetic, is a laborious process; or of the extraction of roots, to which the common rules furnish but a circuitous and slow approximation.

The only instruments which promise to afford real assistance to the practical calculator, are those founded on the theory of logarithms: a theory, which has been the fertile source, as well as the commodious instrument of discovery in every department of mathematical inquiry. The scale of Gunter, and the common sliding-rule, are derived from the properties of logarithms; and the purposes to which they are immediately applicable are the *multiplication* and *division* of numbers. The instrument, of which I purpose giving an account to the Society in this communication, is founded on a particular mode of employing logarithms, and is calculated to apply immediately to the *involution* and *evolution* of numbers. To those who are already conversant with mathematical pursuits, a few words would suffice to explain the principle on which it operates: but to such as are not familiar with the practical employment of logarithms, or of the common sliding-rule, the following statement of the chain of reasoning on which they depend, may conduce to render the subsequent details more intelligible.

The mode in which logarithms are instrumental in facilitating computation, is by converting the more difficult and laborious into the simpler operations of arithmetic. This is effected by substituting, instead of the numbers on which the requisite operations are to be performed, other numbers previously calculated and arranged in tables, so that every number in the natural series, has one of these artificial numbers, or logarithms, corresponding to it. These logarithms are so calculated, that, by adding together those which correspond to any given factors, the sum obtained shall be the logarithm, or artificial number, corresponding to the product of these factors. By consulting the tables, therefore, this product may be discovered; for it will be the number answering to the logarithm, or sum, thus obtained. The subtraction of one logarithm from another will, in like manner, give the logarithm of the quotient resulting from the division of the number corresponding with the second, by the number corresponding with the first. The multiplication of a logarithm by any number will change it into another logarithm, which will answer to that power of the number corresponding with the former logarithm, which has this multiplier for its exponent.

But it will be seen that even in the simplest and most direct applications of this invention some exertion of arithmetical skill and some share of mathematical knowledge are requisite. Even this species of labour may, however, be avoided by the employment of lines as the representatives of logarithms; so that by the simple admeasurement of these lines, with their sums, differences, or multiples, on a given scale, the result of any of the above mentioned operations may be obtained within a certain degree of accuracy. A farther improvement consists in graduating a line of a convenient length *logometrically*, that is, dividing it so that the distance of each division from the beginning of the line, which is marked with unity, shall measure, on a given scale of equal parts, the logarithm of the number which is affixed to it. A line so divided is known by the name of GUNTER's scale.

The divisions which are situated at equal distances, being marked by numbers whose logarithms have equal differences, it follows that the spaces intervening between any two numbers are proportional to the differences between their respective logarithms; or are measures of the ratios between each of these numbers. The same use may therefore be made of such a scale, as of a table of logarithms with regard to operations to be performed on their corresponding numbers. Thus it will be found, that the portion of the scale extending from 1 to 3, added to that extending from 1 to 4, is equal to that between 1 and 12: showing that the logarithm of 3, added to that of 4, is equal to the logarithm of 12: or that the ratio of 1 to 3, added to that of 1 to 4, composes the ratio of 1 to 12: or that 12 is the product of 3 and 4. The excess of the interval between 1 and 24, over that between 1 and 6, or, what is the same thing, the interval between 6 and 24, will be equal to that between 1 and 4; showing, by a similar process of reasoning, that 4 is the quotient of 24 divided by 6. This comparison of the intervals between the numbers on GUNTER'S scale is effected with great ease by the addition of another scale, which may be called the slider, exactly equal in length to the former, and bearing the same divisions, but capable of being moved by its side, so as to allow of any part of the one being applied to any part of the other. In this form it constitutes the common sliding-rule, the utility of which is so generally known in resolving all questions that require the simple operations of multiplication and division, or relate to the finding of any term of a proportion. Supposing the two scales originally to coincide, the sliding scale being the undermost, by advancing the slider any given distance, each of its divisions will be brought under those of the fixed scale, which before were respectively situated farther forwards by an interval equal to that given distance. Every number in the upper scale will therefore have to the number standing under it on the slider, the same constant ratio; a ratio indicated by the number under which the unity, or commencement of the scale, of the slider has been placed. The former numbers will therefore be the multiples of the latter by this constant number. Thus, by adjusting the slider, so that its unity shall stand under any given multiplier or divisor, the upper line will exhibit the series of the products of all the subjacent numbers by the given multiplier: and conversely, the slider will exhibit the series of the quotients resulting from the division of the numbers immediately above them by the given divisor.*

* As the practical mode of using the sliding-rule is frequently not obvious even to those who are in possession of the principle of its construction, I shall beg leave to point out the following proposition, as one that leads directly to the solution of every case to which the instrument can be applied, and an attention to which, therefore, may conduce to its more ready and more general employment. *In every position of the slider, all the fractions formed by taking the numbers on the upper line as numerators, and those immediately under them as denominators, are equal.* Thus every corresponding numerator and denominator, having to each other the same ratio, may be considered as two terms of a proportion. Any two of these equivalent fractions will therefore furnish the four terms of a proportion; of which any unknown term may be supplied, when the others are given, by moving the slider till the numbers composing the terms of the given fraction, are brought to coincide on the two lines. The required term will then be found occupying its proper place opposite to the other given term. Thus, from the proportion A : B : : C : D, we may derive $\frac{A}{B} = \frac{C}{D}$; and adjusting the slider so that B shall stand under A, D will be found under C, when C is given: or C will be found over D, when D is given. A similar process would have furnished A or B, when one of them together with C and D, were given. Since the products of each numerator by the denominator of the other fraction are equal; (that is, AD = BC); when one of the terms is unity, the question becomes one of simple multiplication or division. The product of A and B, which we may call P, will be found, as before, by placing the slider so as to express the fractions $\frac{A}{1} = \frac{P}{B}$. The quotient of A divided by B, which we may call Q, will in like manner be found by forming the fractions $\frac{A}{B} = \frac{Q}{1}$: that is, in the former case, the product P will stand over B, when the 1 on the slider is brought under A; and in the latter case, the quotient Q will stand over the 1 of the slider when B is brought under A.

This instrument has been variously modified with a view of enlarging its scale, or of adapting it to particular objects, such as the calculation of exchanges, the measuring of plane and solid bodies, and the computations of trigonometry. The Society has recently witnessed its successful application, by Dr. WOLLASTON, to another science, in his synoptic scale of chemical equivalents, for the invention of which every practical as well as philosophical chemist must acknowledge to him their deep obligation.

But to whatever purposes the sliding-rule may have been applied, its use is necessarily limited to those operations which are performed by the simple addition or subtraction of logarithms, and to the corresponding arithmetical operations above mentioned. It is not directly adapted to the multiplication or division of logarithms by any number, and therefore is not directly calculated to perform the involution or evolution of numbers, to which, as was before noticed, the multiplication and division of logarithms correspond. Yet many practical, as well as philosophical, inquiries occur, in which it is necessary to ascertain the powers and roots of numbers. In all researches, for example, which involve geometrical progressions, or exponential quantities, and whenever the terms of a series are to be computed in obtaining approximate solutions, these questions present themselves. The common sliding-rule furnishes no direct mode of determining even the simple power or root of a given number: and when the exponent of the required power or root is not an integral, but a fractional number, its inadequacy to resolve the question is still more apparent. The squares and square roots, it is true, are often pointed out on the common rules, by means of a supplementary line graduated so that each of its divisions are double in length to those of the two other lines. A line of cubes, or cube roots, or of any other given power, might, in like manner be subjoined. But it is obvious that the uses of any such additional lines are confined to cases where a particular power is concerned: they give us no assistance in the case of any other power or root, which has no immediate relation with the former.

A new mode of graduation has occurred to me which possesses these requisites, and exhibits, on simple inspection, all the powers and roots of any given number, to any given exponent, with the same facility, and in the same way, that products, quotients, and proportionals, are exhibited by the common sliding-rule. It is accordingly a measure of powers, in the same way as the scale of GUNTER is a measure of ratios. An example will best illustrate the principle of its construction. If it were required to raise the number 2.123 to the fifth power: availing ourselves of logarithms, we should multiply the logarithm of 2.123 (or 0.32695) by 5. The product (1.63475) would be found by the tables to correspond to 43.127, which, with decimals to three places only, is the number required, or the fifth power of 2.123. If the exponent, instead of an entire number, as 5, were fractional, as 4.3719, the operation of multiplying by such a number would be more tedious, and might evidently again be abridged by having recourse to logarithms. Taking, then, the logarithm

of - - -	0.32695 or	9.5144813
and the logarithm of -	4.3719 or	0.6406702
and adding them, we obtain -		0.1551525

a logarithm answering to the number 1.4294, the product we have been seeking. But this product is itself a logarithm, namely, the logarithm of the power required. The number having for its logarithm 1.4294, namely, 26.878, is therefore the power sought for, or $2.123^{4.3719}$.

It may be observed, in this last example, that of the numbers added together, the first was the logometric logarithm, (that is, the logarithm of the logarithm) of the given root: the second was the simple logarithm of the exponent; and the sum of these was the logometric logarithm of the power. If, therefore, we were at the pains to construct a table having three sets of columns; the first containing the natural series of numbers; the second, their corresponding logarithms; and the third, containing the logarithms of those logarithms; we should possess the means of raising any given number to any given power, by the simple addition of the numbers in the second and third columns; just as common multiplications are effected by the addition of common logarithms. It is evident that a line might be graduated so that its divisions should correspond to the numbers in the third column, or should represent the logometric logarithms of the numbers marked upon them: and if this line were applied so as to slide against another line logometrically divided, it would enable us to effect the very operation I have been describing, and thus give us, by inspection, the powers corresponding to any given root and exponent.

The instrument, then, in its simplest form, would consist of two graduated scales applied to each other. A portion of these scales is represented, Pl. II, fig. 1. The lower rule, AA, which I shall call the slider, is the common GUNTER'S double line of numbers, or is a line logometrically divided; the divisions of the first half being from 1 to 10, and being repeated on the second half in the same order. The upper or fixed rule, BB, is graduated in such a manner, that each of its other divisions is set against its respective logarithm on the slider; and, consequently, all the numbers on the slider will be situated immediately under those numbers in the upper rule, of which they are the logarithms. Thus 2 on the rule will be over 0.30103 of the slider; 3 over 0.47712: 2 on the slider will stand under 100 on the rule; 3 under 1000; and so on.

As the series of ordinary logarithms express the exponents of 10, of which the corresponding numbers are so many successive powers, it is evident that, in this position of the instrument, the upper line will exhibit the series of the powers of 10, corresponding to all the exponents marked on the slider. It will be seen, for instance, that the second power of 10 is 100, the third, 1000, &c.: that the 0.5th (or the square root) is 3.163; the 0.25th (or the fourth root) is 1.778; the 0.2th (or the fifth root) is 1.585: and so on.

In every other position of the slider, the upper rule will exhibit, in like manner, the series of powers of that number under which the unit of the slider has been placed, while the opposite numbers on the slider are the exponents of those powers. Thus, if (as in Pl. II, fig. 2) the unit of the slider be placed under the division 3 of the upper rule (at R); the square of 3, or 9, will be found over the 2 of the slider: its cube, 27, over the 3; its fourth power, 81, over the 4: and so on for any other power. It is evident, then, that in order to find a given power of any number, the unit of the slider must be set underneath that number in the upper rule; and that the number sought will then be found above that number in the slider which expresses the magnitude of the required power.

Such being the mode of its application to the finding of powers, its use will be obvious in performing the contrary operation of finding roots. The root might, for this purpose, be considered as a fractional power: but as this would require a reduction to decimals, the easiest mode will be to place the number expressing the degree of the required root under the given number, and the root itself will then be found over the unit, or beginning of the scale, in the slider. For fractional powers, the denominator of the exponent must be placed under the root, and its numerator will then point out the power.

It is hardly necessary to add, that by the same mode we may discover the exponent of any given power to any given root: since, whatever be the root over the unit of the slider, the whole series of the powers of that root, with their corresponding exponents, are rendered apparent. This circumstance may indeed be considered as an additional recommendation to the employment of this instrument: for it affords to those less versed in the contemplation of numerical relations an ocular illustration of the theory of involution. It presents, at one view, the whole series of powers arising from the successive multiplication of all possible numbers, whether entire or fractional; and exhibits this series in all its continuity when the exponents are fractional, and even incommensurate with the root itself. The production of the upper line in one direction conveys a more accurate notion of the progressive and rapid increase of those powers, than can be acquired by mere abstract reflection: and its continuation on the other side, shows the slow approximation to unity which takes place in the successive extractions of higher and higher roots.

A variety of forms of construction might be given to instruments operating on the principle now explained. The following has appeared to me, on the whole, to be the most convenient for practical purposes; it is represented on a reduced scale in Pl. II, fig. 3. In order to preserve a sufficient magnitude of scale, I have divided the line of roots and powers into two parts; placing the one above and the other below, and interposing a slider with a double scale of exponents. The slider of the common sliding-rule is graduated in a way that is exceedingly well suited to this purpose, having divisions on each edge, and carrying two sets of numbers from 1 to 10. Adapting a blank ruler to one of these sliders, which must be fixed in a proper position, I mark off, on the upper line, the series of numbers against their respective logarithms on the slider; placing 10 over the middle unit of the slider, 100 over the 2, 1000 over the 3, and so on, proceeding towards the right from 10 to 10000000000, the tenth power of 10, an extent which is more than sufficient for all useful purposes. The space to the left is also graduated on the same principle, from 10 to 1.259 which is the tenth root of 10, or $10^{0.1}$. The upper portion of the rule being thus filled, I place the continuation of the same line on the lower portion, beginning on the right hand, and proceeding in a descending series of fractional powers of 10, corresponding with the exponents on the intermediate slider, which, when applied to this portion, are to be taken as only one hundredth of their value when applied to the upper portion. While 1.259 therefore is marked on the right, 1.0233 $= 10^{0.01}$) will occupy the middle, and 1.002305 ($= 10^{0.001}$) the left end of the lower line. It is evident that the graduation might thus be continued indefinitely in both directions. But for all practical purposes the limits thus obtained will be found amply sufficient: for the well known property of the logarithms of roots in a descending series, enables us to dispense with all farther continuation of the scale in that direction. In proportion as numbers in a descending series approach very near to unity, their logarithms bear more and more exactly a constant ratio to the excess of those numbers above unity, namely, the ratio expressed by the modulus of the system, or 1 to .4342944819, &c. As we descend in the scale, therefore, the decimal part of the exponents becoming smaller and smaller, the corresponding logarithms will approximate so nearly to the multiple of that decimal part by this modulus, that no *sensible* error will result from assuming them to be the same.* The divisions to the left of the lower portion of the rule may therefore be taken as sufficiently accurate representations of the divisions which would occur in the succeeding portions of the line, if it were prolonged indefinitely in that direction.

The applications of which this instrument is susceptible are

* Thus the logarithm of 1.05 is .021189
that of 1.005 is .0021661
and of 1.0005 is .00021709
which differs from the product of the modulus by .0005 (or .00021715) by a quantity affecting only the fourth significant figure. The roots 1.0005, 1.00005, 1.000005, &c. may, therefore, without sensible error, be considered as coinciding with the division 217 on the slider.

various, and will easily present themselves. In many speculative and practical inquiries, cases occur in which geometrical progressions are concerned, and in which it becomes a question, the first term and the common ratio being given, to find the other terms; or, knowing the first and also any other term, to ascertain the rate of increase. In all these cases, it is obvious that the first term is to be regarded as the root, or first power, and the unit in the slider adjusted, so as to coincide with that number in the line of powers, that is, in the upper and lower portions of the fixed rule. The number of terms will constitute the exponent of the series, and the power corresponding to each successive exponent 2, 3, 4, &c. will be the second, third, fourth, &c. term of the progression.

The successive amounts of a sum placed at compound interest compose a geometrical progression; and accordingly all questions of compound interest are resolvable by this instrument. The rate of interest, or the per centage per annum, being added to 1, gives the amount of £1. at the end of one year. Thus, at 5 per cent. the amount is 1.05, at 3 per cent. 1.03, and so on. In either case this number is to be regarded as the first term, or root of the series. Setting the unit of the slider against this number on the rule, we shall find the amount of £1. at the end of $5\frac{1}{2}$ years, opposite to the number 5.5 on the slider, and the same of any other interval of time. If it be required to ascertain in what time a sum placed at compound interest at 3 per cent. would be doubled: placing the unit over 1.03, the number 2 on the rule will indicate 23.45 on the slider, as the number of years required for doubling the sum at that rate of compound interest.

Questions relating to the increase of population and to the calculation of chances, involve the investigation of powers, which may be facilitated by this instrument. Examples of its application also occur in considering the reduction of temperature which bodies undergo by the communication of heat to surrounding bodies, the quantities of light transmitted through different thicknesses of a transparent medium: the diminution of density which the air in a receiver undergoes during its exhaustion by the air pump, and the relation of the density of the atmosphere with its elevation. The interpolation of a given number of mean proportionals between two given numbers, is sometimes required for the solution of a problem, and is easily effected by the rule above described. Thus, in dividing the musical octave into twelve equal semi-tones, the following series of numbers must be calculated, viz. $2^{\frac{1}{12}}$, $2^{\frac{2}{12}}$, $2^{\frac{3}{12}}$, $2^{\frac{4}{12}}$, $2^{\frac{5}{12}}$, $2^{\frac{6}{12}}$, $2^{\frac{7}{12}}$, $2^{\frac{8}{12}}$, $2^{\frac{9}{12}}$, $2^{\frac{10}{12}}$, $2^{\frac{11}{12}}$: this can readily be done in one position of the slider, for when the 12 marked on it is placed under 2 on the rule, the 1 of the slider will point to $1.0595 = 2^{\frac{1}{12}}$, the 2 of the slider will indicate $1.1225 = 2^{\frac{2}{12}}$, the 3, $1.1892 = 2^{\frac{3}{12}}$, &c.

When the first term of a progression is less than unity, all the succeeding terms, that is, all the powers of that fraction, continually decrease. Now all the numbers contained on the rule are above unity: but the terms of such a decreasing progression may yet readily be found by assuming, instead of the first term, its reciprocal, which, being above unity, will of course be contained on the scale. The powers of this reciprocal, will, in like manner, be the reciprocals of the required series, which will accordingly be determined without difficulty. Let the following question, for example, be proposed.

Assuming, that when light is transmitted through water, one half of the quantity that entered is lost by passing through seven feet of water:* how much will be intercepted by passing through three feet? In questions of this sort it must be recollected that it is the quantities of transmitted, and not of intercepted light, that are in geometrical progression. If 0.5 is transmitted by seven feet, $0.5^{\frac{3}{7}}$ will be transmitted by three feet. As 0.5 is not contained on the rule, we must take its reciprocal 2, of which the $\frac{3}{7}$th power, or 1.3023, is given by the instrument: this number being opposite to the 3 on the slider, when its division 7 is placed under 2 on the rule. The reciprocal of 1.3023 or .9892 is the quantity transmitted; and therefore .0108 the quantity absorbed by three feet of water.

A variety of propositions relating to the general theory of logarithms are illustrated by this instrument. The assumption of the number 10, as the basis of our system of logarithms is arbitrary, and is chosen only for the sake of greater convenience in computation. The hyperbolic system, which has the number 2.302585093, &c. for its basis, possesses other advantages, especially in the higher branches of analysis. The instrument may be made to exhibit at one view the series of any particular system of logarithms, that is, of a system with any given basis, or any given modulus, by merely setting the unity of the slider against the given basis on the rule: or the given modulus on the slider against the number 2.7182818, &c. on the rule. The divisions on the slider will then denote the logarithms of the numbers opposed to them on the rule.

* YOUNG's Lectures on Natural Philosophy, I. 409.

Let it be required to determine the particular system of logarithms, in which the modulus shall be equal to the basis. Take out the slider, and introduce it in an inverted position, so that the numbers on it shall increase from right to left: and place the number .4343, &c. (the modulus of the common system) under 10 (its corresponding basis) on the rule, as represented in Pl. II, fig. 4. We shall find that in this position, all the other numbers on the slider will be the moduli corresponding to the respective bases of each different system, on the rule. Thus, the 1 on the slider, or the modulus of the hyperbolic system, is opposite to 2.718, the basis of that system. On the other hand, the division 2 on the rule is opposite to 1.4427, which is the modulus of the system having for its basis the number 2. Carrying the eye still more to the left, and observing the point where similar divisions appear both on the rule and the slider, we shall find it to be at the number 1.76315, which therefore expresses the modulus and the basis in that particular system in which they are both equal. The reason of the above process will readily appear when it is considered, that the modulus of every system is the reciprocal of the hyperbolic logarithm of its basis.

This inverted condition of the slider will also afford an easy method of solving exponential equations, for which there exists no direct analytical method. The following may serve as an example. Let the root of the equation $x^x = 100$ be required. Set the unit of the inverted slider under 100 on the rule, and observe, as before, the point where similar divisions coincide; this will be at 3.6, which is a near approximation to the required root: and accordingly $3.6^{3.6} = 100$.

The principle of the instrument above described might be applied in a variety of different forms to these several purposes: and I shall beg leave to notice one or two that offer some peculiarities. If to the upper scale, which we may suppose to be fixed, and to be graduated logometrically, constituting, as we have already seen, the line of exponents, a slider be adjusted, graduated on both edges, according to the logometric logarithms; and the line below, which like the upper one is supposed to be fixed, be graduated in the same manner as the slider, the instrument will possess the following property. When the division 10 on the slider is set against any particular number, or exponent, in the upper line, all the numbers on the lower line will be the powers, to the same degree, of the numbers opposite to them on the slider: the degree of the power being marked by the exponent on the upper line which is above the 10 on the slider. The lower line, therefore, will exhibit the whole series of similar powers belonging to all possible roots; and conversely, the slider will exhibit all the roots of the same dimension, with regard to all possible numbers. Thus, if the 10 on the slider be under 3 in the line of exponents, it will itself be above 1000 (which is its cube) in the lower line; all the other numbers in that line will be the cubes of their opposites on the slider; and, conversely, the former will every where be the cube roots of the latter. This will be sufficiently apparent, when it is recollected that the addition or subtraction of logometric logarithms answer to the multiplication or division of simple logarithms, and therefore to the involution and evolution of numbers. The rule in this form, therefore, bears a closer analogy to the common sliding-rule; since in every position it exhibits the series of similar powers and roots, exactly in the same way as the latter exhibits the series of similar products and quotients.

I have also contrived another form of the instrument which possesses some advantages in theory, though its execution may perhaps be more difficult. It is evident that the whole scale may, like GUNTER'S line, be thrown into a circular form; and this I have done in the way represented in Pl. III. The circle on the outside, being logometrically divided from 1 to 10 round the circumference, will constitute the line of exponents. The line of powers, being disposed in a spiral, will occupy the interior space, which may be made to revolve within the former, and should be provided with one or more threads extending from the centre to the circumference, and serving as radii to mark the position of all the parts of the spiral line with regard to the divisions of the outer circle. One of these threads may be fixed at the unit or beginning of the scale, and will serve to mark the position for the root of any required power. The spiral itself must be graduated exactly as the upper line in the first described rule: that is, the situation of the division 10 must be first determined upon, and then brought under the unit in the circle of exponents, that is, under the fixed thread. Every other division must then be marked with reference to the place of its logarithm on the circle, or must be made to occupy the same angular distance from the thread. This graduation will be most conveniently made by means of the moveable leg of a sector revolving on the centre of the circle. The comparison of the divisions of the spiral with those of the circle, may be made, either with this moveable sector, or with the threads already mentioned. The numbers on the spiral will increase as they recede from the centre, and each turn will carry on the powers to an exponent 10 times higher than the preceding: and the converse will obtain with regard to the descending portion. Thus, immediately in a line with the 10, on the superior branch of the spiral, is found the number 10000000000, or 10^{10}: below it on the inferior branches, we find successively $1.258926 = 10^{0.1}$, $1.023293 = 10^{0.01}$, $1.00230524 = 10^{.001}$, $1.000230285 = 10^{.0001}$, $1.0000230261 = 10^{.00001}$, &c. of which, agreeably to the remarks that were formerly made, the decimal figures approach nearer and nearer to 2.302585093, &c. the reciprocal of the modulus of the logarithmic system.

A much greater extension might be given to the scale, by multiplying the number of turns of the spiral corresponding to the decuple increase of the exponents: but the superior accuracy thus obtained would probably be overbalanced by the diminished conveniency of application.

It is possible to exhibit in one view the whole series of roots, powers, and exponents, in all their possible relations, by the following disposition of lines. Let the lines AB, AC, (Pl. IV.), which I shall call respectively the line of exponents, and the line of roots, be drawn at right angles to each other, and a diagonal AD, or line of powers be drawn, bisecting this angle. Divide AB logometrically, so that the unit of the scale shall be at A: upon the same scale, divide AC into logometric logarithms, and AD into similar parts, by perpendiculars from the divisions of AC.

Through all these points of division, let there be drawn perpendiculars to the respective lines: and let each of these perpendiculars be considered as referring always to the numbers on the lines from which they are drawn. The following relation will be preserved between the numbers belonging to any three of these perpendiculars that meet in one point; viz. that the number of the perpendicular to the line of roots, raised to the power expressed by the perpendicular to the line of exponents, shall be equal to the number denoted by the perpendicular to the line of powers. To find, for example, the second power of 3; following the perpendicular from the division 3 on the line of roots, and that from the division 2 on the line of exponents, till they meet in the point *e*, we find among the other set of perpendiculars, the line *fg* passing through the same point, which, followed till it meets the line of powers, indicates on it the number 9, which is accordingly the second power of 3. A similar process in other directions will furnish the root when the power and exponent are given, or the exponent when the root and power are given. We may thus perceive, at a single glance, not only all the powers of any particular root, and all the roots of any particular power; but also all the exponents of the series of powers belonging to the same root, as well as the similar powers of every possible root.

It is, perhaps, superfluous to observe, that the same method is applicable to the common scale of Gunter; and that a table constructed accordingly, by dividing the sides as well as the diagonal logometrically, and applying three sets of perpendiculars, would, by their intersections, exhibit in one view all possible products and quotients resulting from all possible factors or divisors.

Fig. 1.

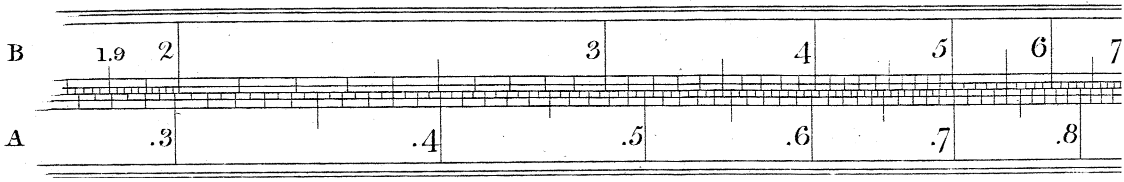


Fig. 2.

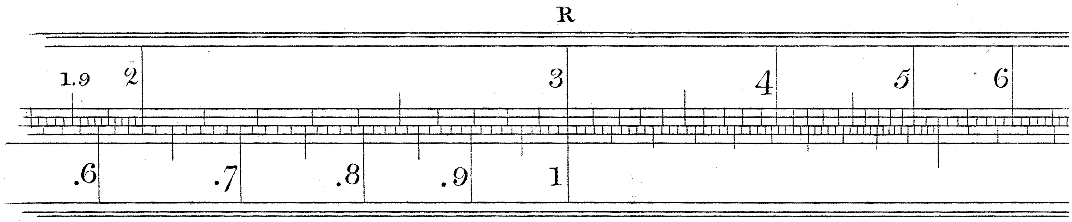


Fig. 3.

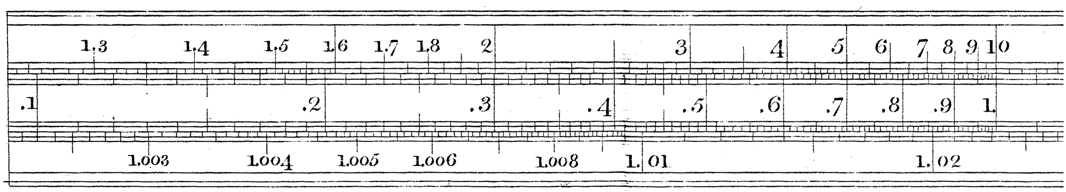
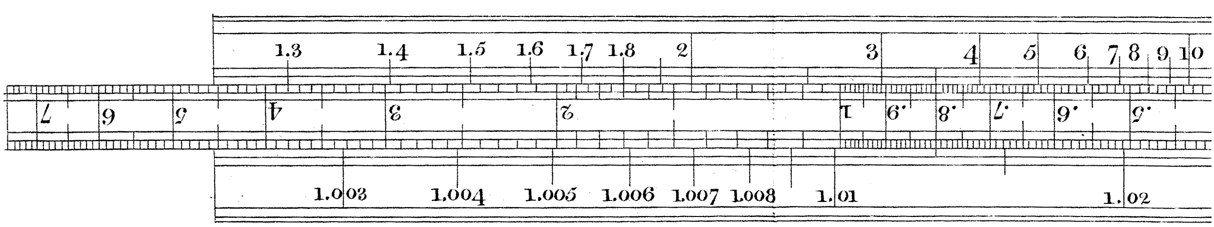
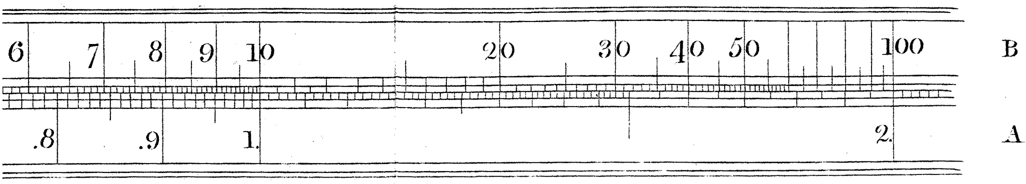


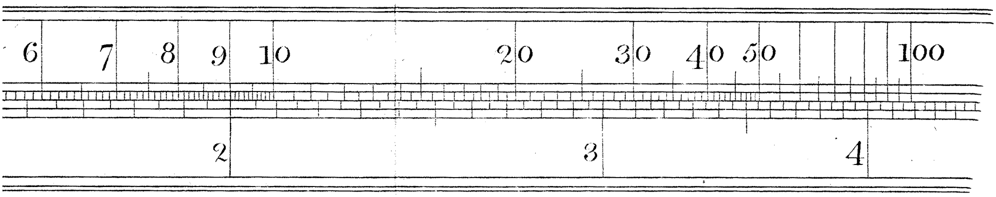
Fig. 4.



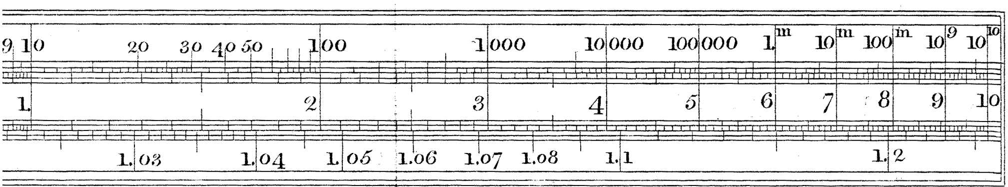
1.


6 7 8 9 10 20 30 40 50 100
B
.8 .9 1. 2.
A

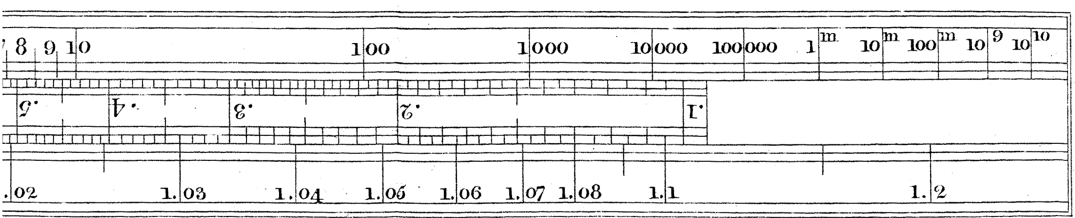
2.


6 7 8 9 10 20 30 40 50 100
2 3 4

3.

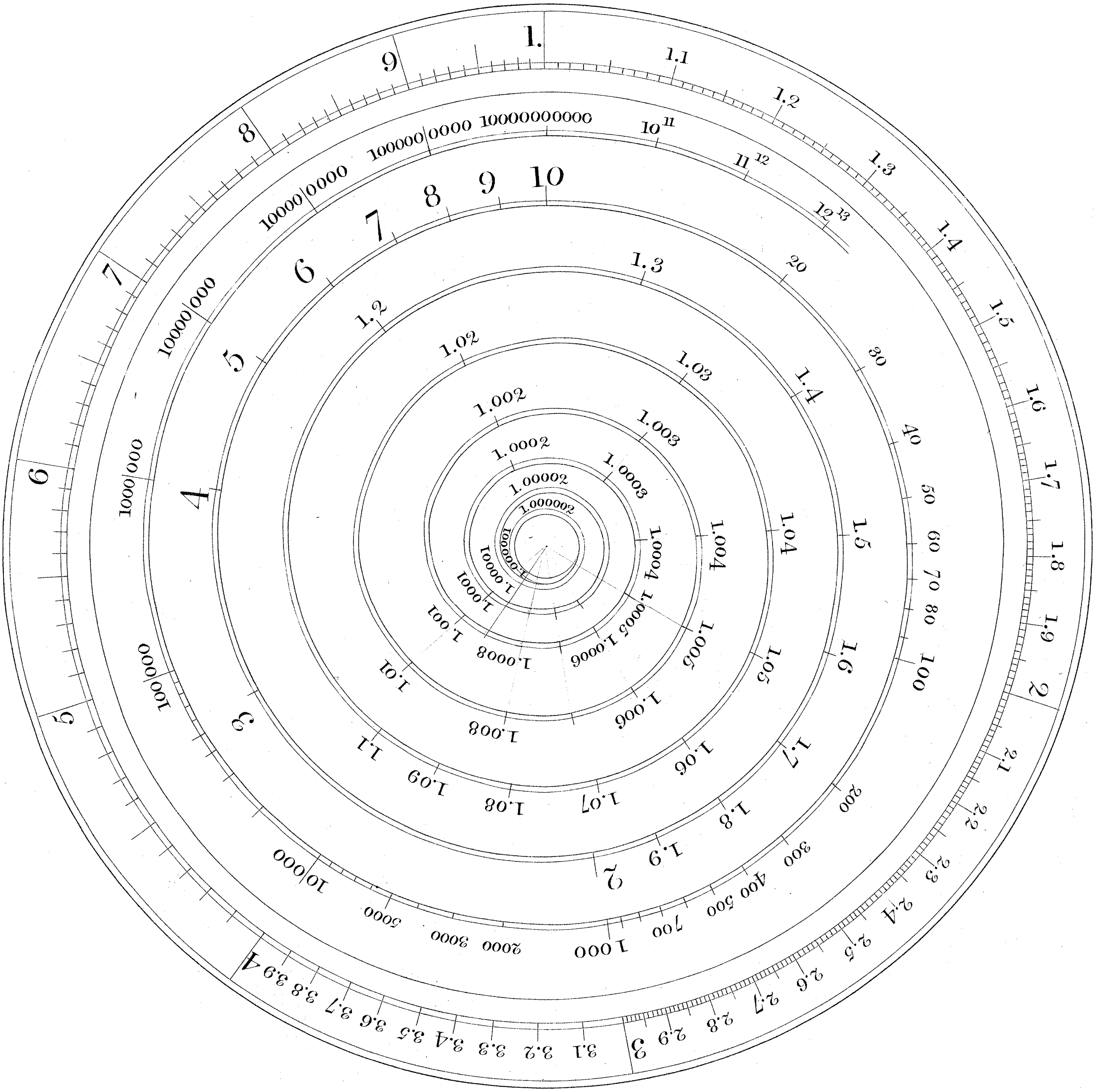

9 10 20 30 40 50 100 1000 10000 100000 1m 10m 100m 10 9 10 10
1 2 3 4 5 6 7 8 9 10
1.03 1.04 1.05 1.06 1.07 1.08 1.1 1.2

4.


8 9 10 100 1000 10000 100000 1m 10m 100m 10 9 10 10
.1 .2 .3 .4 .5
.02 1.03 1.04 1.05 1.06 1.07 1.08 1.1 1.2

Js. Basire sc.

Fig. 5.



J. Basire, sc.

Fig. 6.

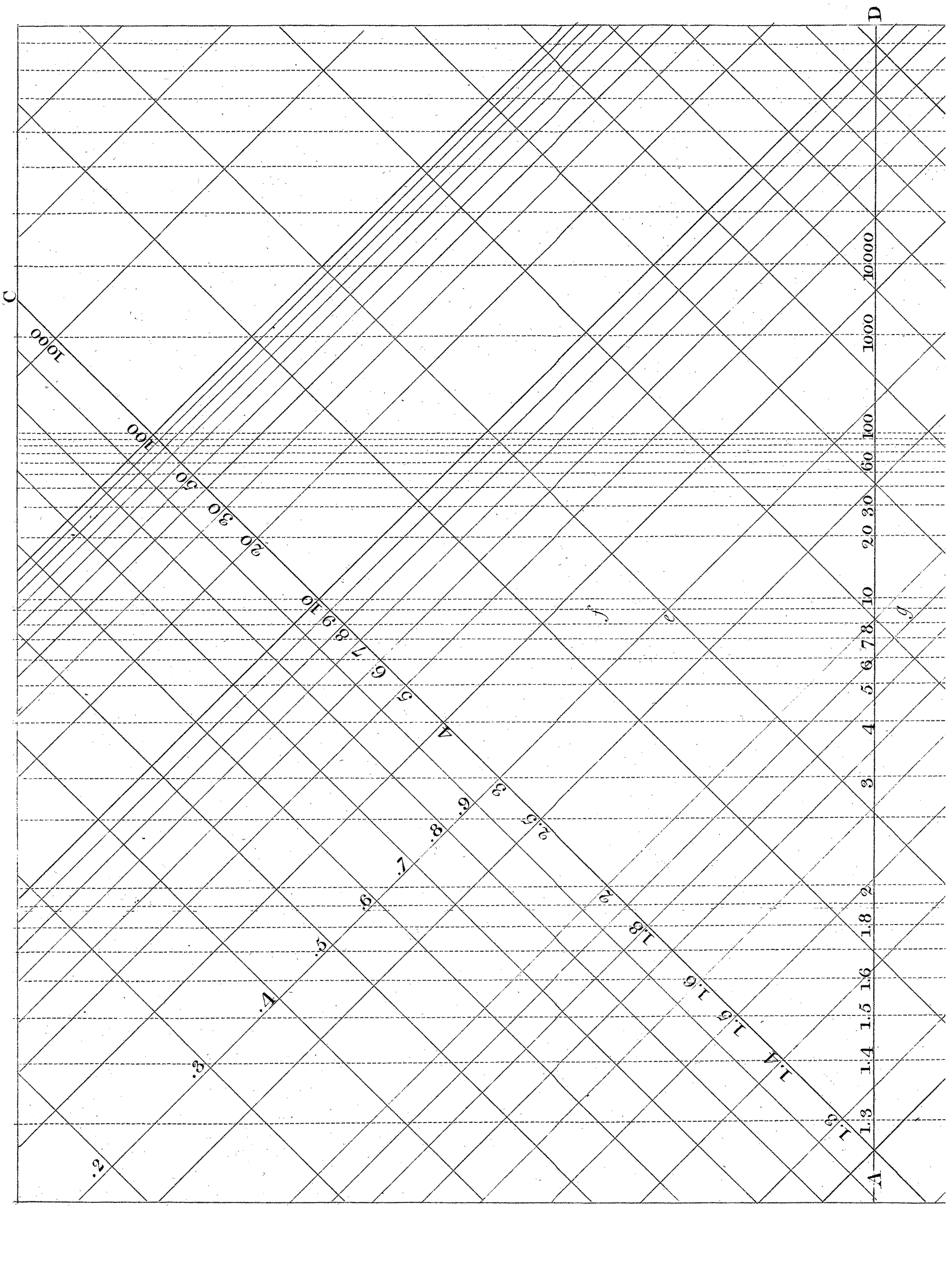


Fig. 6.

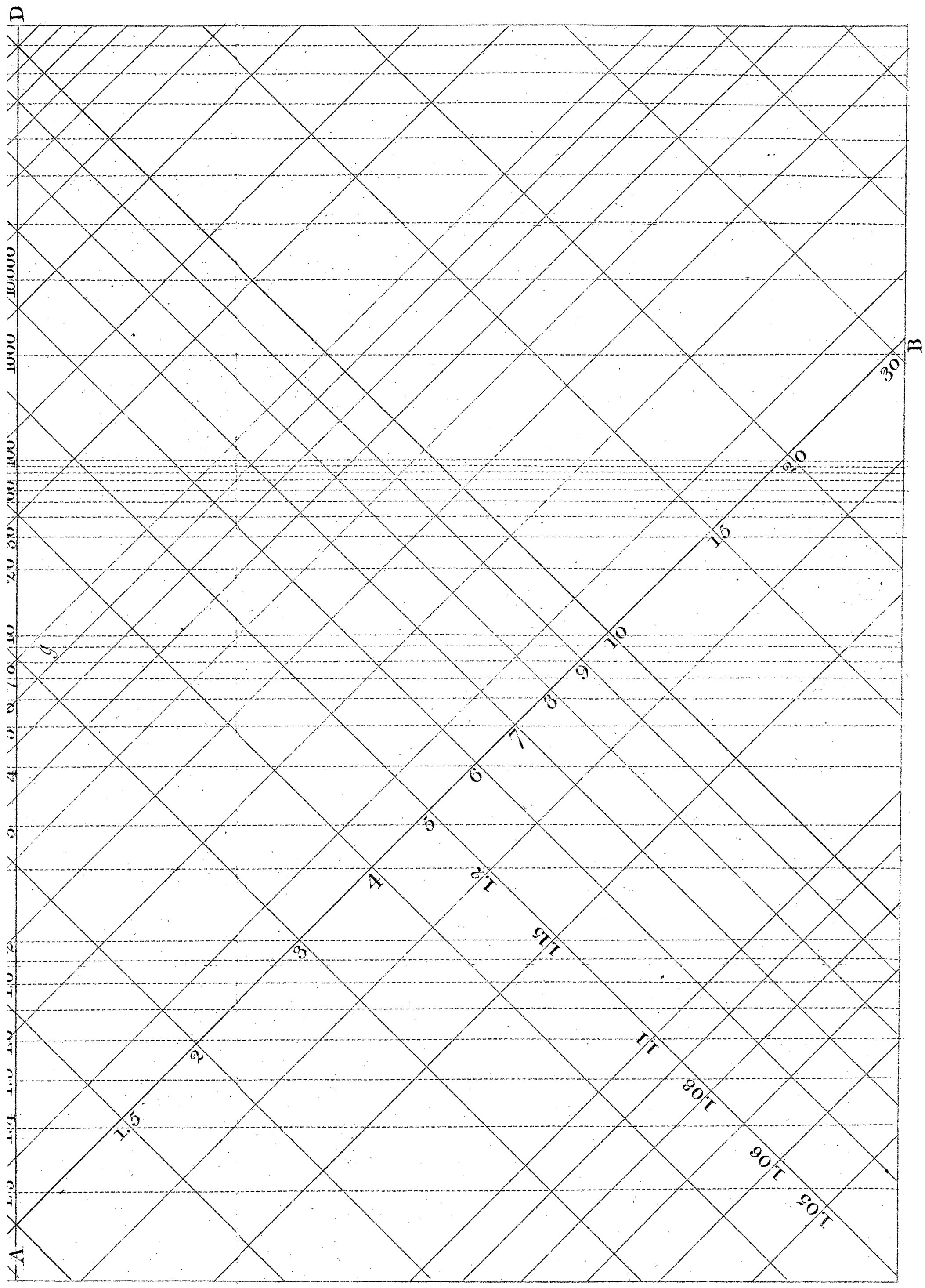


Fig. 1.

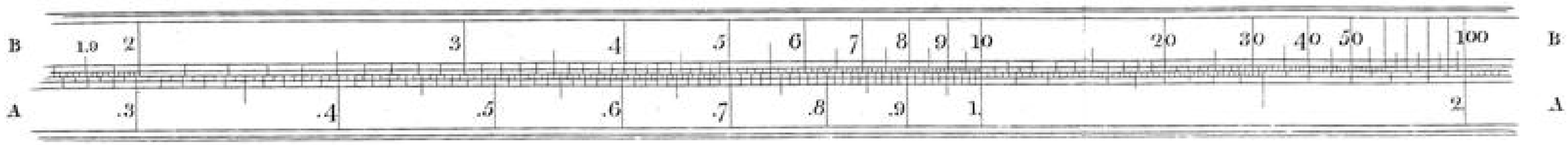


Fig. 2.

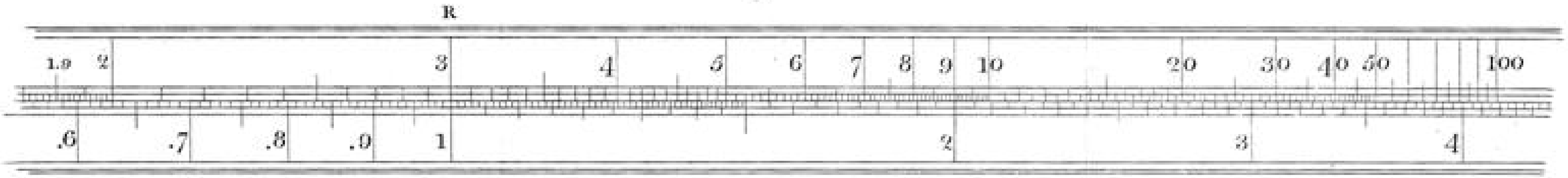


Fig. 3.

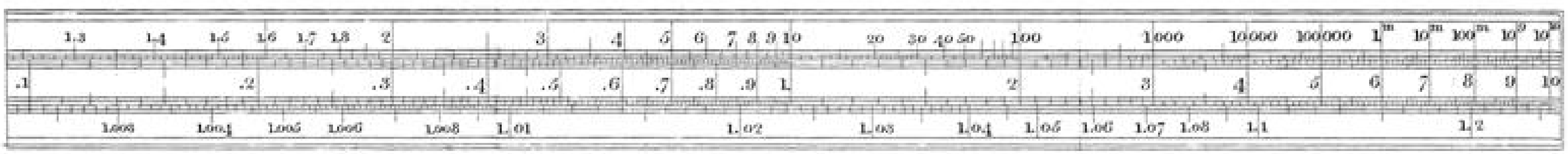
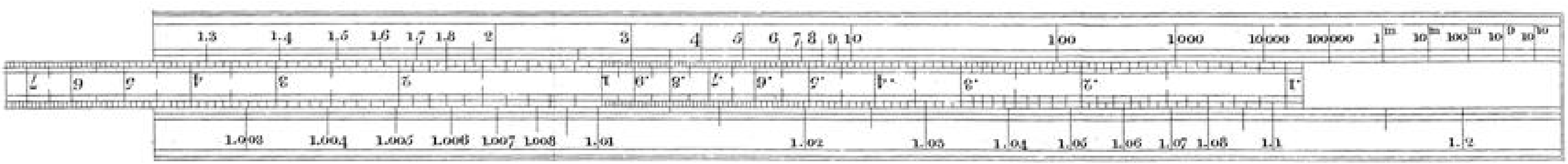


Fig. 4.



J. Basire sc.

Fig. 6.